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LETTER TO THE EDITOR

Classification of subalgebras in the symplectic model

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Abstract. An so(3) tensor realisation for the Lie algebra sp(3, R) of the symplectic nuclear collective model is given. In this realisation, a complete classification of all subalgebras of sp(3, R) that contain the physical angular momentum algebra so(3) is obtained.

The symplectic shell model provides a natural framework for the simultaneous macroscopic and microscopic description of nuclear quadrupole collective dynamics. An excellent review article on this subject has been published recently (Rowe 1985), and we shall adopt the notation of this paper here. The rich structure of the symplectic shell model is described by its dynamical group Sp(3, R). Recently, Moshinsky (1986) asked whether a complete analysis could be given of the subalgebras of sp(3, R) containing the physical so(3) subalgebra. It is the aim of this letter to provide an answer to this question. We have chosen a simple and straightforward method. First, the sp(3, R) basis elements are realised in terms of so(3) tensor operators. Then it is obvious that for any subalgebra of sp(3, R) that contains so(3), a basis of so(3) tensor operators must exist. Hence, one can systematically consider all subspaces spanned by a set of complete tensor operators and investigate if the space actually forms a Lie algebra.

A common basis for sp(3, R) is given by the six cartesian quadrupole moments (Q_{ij}), the nine gl(3, R) generators of deformations and rotations (S_{ij}, L_{ij}) and the six components of the quadrupole flow tensor (K_{ij})

$$\begin{aligned} Q_{ij} &= \sum_s x_{si}x_{sj} \\ S_{ij} &= \sum_s (x_{si}p_{sj} + p_{si}x_{sj}) \\ L_{ij} &= \sum_s (x_{si}p_{sj} - x_{sj}p_{si}) \\ K_{ij} &= \sum_s p_{si}p_{sj} \end{aligned} \tag{1}$$

where x_{si} and p_{si} ($i = 1, 2, 3$) are the components of position and momentum of the nucleon and Σ_s indicates the summation over all nucleons of the system. In the rest of this letter, the summation Σ_s will be suppressed, since it has no influence in the

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classification of subalgebras. Hence $Q_{ij} = x_i x_j$, $L_{ij} = x_i p_j - x_j p_i$, etc. The commutation relations of the operators (1) are determined by

$$[x_j, p_k] = \mathrm{i}\delta_{jk}. \tag{2}$$

It is convenient to define raising and lowering operators

$$b_j^+ = (1/\sqrt{2})(x_j - \mathrm{i}p_j) \qquad b_j = (1/\sqrt{2})(x_j + \mathrm{i}p_j) \tag{3}$$

satisfying

$$[b_j, b_k^+] = \delta_{jk} \tag{4}$$

and giving rise to a new basis for sp(3, R):

$$A_{ij} = b_i^+ b_j^+ \qquad B_{ij} = b_i b_j \qquad C_{ij} = \tfrac{1}{2}(b_i^+ b_j + b_j b_i^+) \qquad (i, j = 1, 2, 3). \tag{5}$$

The operators C_{ij} span the u(3) subalgebra of sp(3, R), and the physical so(3) subalgebra is spanned by the L_{ij}, or equivalently by $\mathrm{i}(C_{ji} - C_{ij})$. We define the spherical components of $b_j^{(+)}$ by

$$\begin{aligned} &\pi_0^+ = b_3^+ \qquad \pi_0 = b_3 \\ &\pi_{\pm 1}^+ = \mp(1/\sqrt{2})(b_1^+ \pm \mathrm{i}b_2^+) \qquad \pi_{\pm 1} = \mp(1/\sqrt{2})(b_1 \mp \mathrm{i}b_2). \end{aligned} \tag{6}$$

The only non-vanishing commutators among these components are

$$[\pi_\mu, \pi_\nu^+] = \delta_{\mu\nu}. \tag{7}$$

The elements π_μ^+ and $\tilde{\pi}_\mu = (-1)^\mu \pi_{-\mu}$ $(\mu = -1, 0, 1)$ form an so(3) tensor of rank 1. By means of these spherical components one defines

$$\begin{aligned} &G_\kappa^{(k)} = [\pi^+ \times \tilde{\pi}]_\kappa^{(k)} + (-1)^k [\tilde{\pi} \times \pi^+]_\kappa^{(k)} \qquad (k = 0, 1, 2) \\ &T_\kappa^{(k)} = [\pi^+ \times \pi^+]_\kappa^{(k)} \qquad U_\kappa^{(k)} = [\tilde{\pi} \times \tilde{\pi}]_\kappa^{(k)} \qquad (k = 0, 2) \end{aligned} \tag{8}$$

where

$$[\pi^+ \times \tilde{\pi}]_\kappa^{(k)} = \sum_{\mu, \nu} \langle 1\mu 1\nu | k\kappa \rangle \pi_\mu^+ \tilde{\pi}_\nu \tag{9}$$

and $\langle \ | \ \rangle$ is an so(3) coupling coefficient (Edmonds 1957). The commutation relations among the basis elements (8) are then expressed by means of $3j$ and $6j$ symbols:

$$\begin{aligned} [G_\kappa^{(k)}, G_{\kappa'}^{(k')}] = \sum_{k'', \kappa''} &(-1)^{\kappa''+1} [(2k+1)(2k'+1)(2k''+1)]^{1/2} \\ &\times \begin{pmatrix} k & k' & k'' \\ \kappa & \kappa' & -\kappa'' \end{pmatrix} \begin{Bmatrix} k & k' & k'' \\ 1 & 1 & 1 \end{Bmatrix} 2[(-1)^{k+k'+k''} - 1] G_{\kappa''}^{(k'')} \end{aligned} \tag{10}$$

$$\begin{aligned} [U_\kappa^{(k)}, T_{\kappa'}^{(k')}] = \sum_{k'', \kappa''} &(-1)^{\kappa''+1} [(2k+1)(2k'+1)(2k''+1)]^{1/2} \\ &\times \begin{pmatrix} k & k' & k'' \\ \kappa & \kappa' & -\kappa'' \end{pmatrix} \begin{Bmatrix} k & k' & k'' \\ 1 & 1 & 1 \end{Bmatrix} 2G_{\kappa''}^{(k'')} \end{aligned} \tag{11}$$

$$\begin{aligned} [G_\kappa^{(k)}, T_{\kappa'}^{(k')}] = \sum_{k''\kappa''} &(-1)^{\kappa''+1} [(2k+1)(2k'+1)(2k''+1)]^{1/2} \\ &\times \begin{pmatrix} k & k' & k'' \\ \kappa & \kappa' & -\kappa'' \end{pmatrix} \begin{Bmatrix} k & k' & k'' \\ 1 & 1 & 1 \end{Bmatrix} 4(-1)^k T_{\kappa''}^{(k'')} \end{aligned} \tag{12}$$

$$\begin{aligned} [G_\kappa^{(k)}, U_{\kappa'}^{(k')}] = \sum_{k''\kappa''} &(-1)^{\kappa''+1} [(2k+1)(2k'+1)(2k''+1)]^{1/2} \\ &\times \begin{pmatrix} k & k' & k'' \\ \kappa & \kappa' & -\kappa'' \end{pmatrix} \begin{Bmatrix} k & k' & k'' \\ 1 & 1 & 1 \end{Bmatrix} (-4) U_{\kappa''}^{(k'')}. \end{aligned} \tag{13}$$

Note that $G_\kappa^{(1)}$ ($\kappa = -1, 0, 1$) are the spherical components of the angular momentum subalgebra so(3). Let $\langle e_1, e_2, \ldots, e_n\rangle$ denote the subspace spanned by $e_1, e_2, \ldots, e_n$; then

$$\begin{aligned} \langle G_\kappa^{(1)}\rangle &= \langle L_{ij}\rangle = \langle C_{ij} - C_{ji}\rangle \\ \langle G_0^{(0)}, G_\kappa^{(2)}\rangle &= \langle C_{ij} + C_{ji}\rangle \\ \langle T_0^{(0)}, T_\kappa^{(2)}\rangle &= \langle A_{ij}\rangle \\ \langle U_0^{(0)}, U_\kappa^{(2)}\rangle &= \langle B_{ij}\rangle. \end{aligned} \tag{14}$$

In order to classify the (real) subalgebras of sp(3, R) containing so(3), we have systematically investigated which so(3) tensors can be put together to form a Lie algebra. The linear combinations of tensor operators that will be considered in the following are supposed to be real combinations. Note that in fact also complex combinations are allowed, as long as all structure constants remain real. The results of our investigation are as follows.

(i) $\quad \mathrm{L}_1 = \langle G_\kappa^{(1)}, aT_0^{(0)} + bU_0^{(0)} + cG_0^{(0)}\rangle.$

Here, a, b and c are arbitrary numbers. This Lie algebra is isomorphic to so(3)$\oplus$u(1).

(ii) $\quad \mathrm{L}_2 = \langle G_\kappa^{(1)}, aT_0^{(0)} + bU_0^{(0)} + cG_0^{(0)}, a'T_0^{(0)} + b'U_0^{(0)} + c'G_0^{(0)}\rangle.$

This forms a Lie algebra if and only if the rank zero tensors (or a linear combination of them) satisfy

$$a'b' - c'^2 = 0 \tag{15}$$

$$a'b + ab' = 2cc'. \tag{16}$$

The Lie algebra L_2 is a semidirect product of the reductive Lie algebra so(3)$\oplus$u(1) with a one-dimensional module.

(iii) $\quad \mathrm{L}_3 = \langle G_\kappa^{(1)}, T_0^{(0)}, U_0^{(0)}, G_0^{(0)}\rangle.$

It is easy to verify that $\langle T_0^{(0)}, U_0^{(0)}, G_0^{(0)}\rangle$ is the basis of sp(1, R). Hence $\mathrm{L}_3 \simeq$ so(3)$\oplus$ sp(1, R). This completes all possibilities of Lie algebras spanned by the 1-tensor $G_\kappa^{(1)}$ and combinations of 0-tensors. Now we systematically bring in 2-tensors.

(iv) $\quad \mathrm{L}_4 = \langle G_\kappa^{(1)}, aT_\mu^{(2)} + bU_\mu^{(2)} + cG_\mu^{(2)}\rangle.$

This always forms a Lie algebra. In order to recognise it, observe that (10)-(12) leads to

$$[aT_\mu^{(2)} + bU_\mu^{(2)} + cG_\mu^{(2)}, aT_\nu^{(2)} + bU_\nu^{(2)} + cG_\nu^{(2)}]$$
$$= \sum_\kappa (-1)^{\kappa+1} 5\sqrt{3} \begin{pmatrix} 2 & 2 & 1 \\ \mu & \nu & -\kappa \end{pmatrix} \begin{Bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \end{Bmatrix} (c^2 - ab)(-4) G_\kappa^{(1)}. \tag{17}$$

Hence there are three cases:

$$\begin{aligned} c^2 - ab > 0 &\Rightarrow \mathrm{L}_4 = \mathrm{L}_4^+ \simeq \mathrm{su}(3) \\ c^2 - ab = 0 &\Rightarrow \mathrm{L}_4 = \mathrm{L}_4^0 \simeq \mathrm{so}(3) \times R^{[2]} \\ c^2 - ab < 0 &\Rightarrow \mathrm{L}_4 = \mathrm{L}_4^- \simeq \mathrm{sl}(3, R). \end{aligned} \tag{18}$$

Only for $c^2-ab=0$ is the Lie algebra not simple, but the semidirect product of so(3) with a five-dimensional module which is a 2-tensor.

(v) $\quad L_5=\langle G^{(1)}_\kappa, aT^{(2)}_\mu+bU^{(2)}_\mu+cG^{(2)}_\mu, a'T^{(0)}_0+b'U^{(0)}_0+c'G^{(0)}_0\rangle.$

There are again three possibilities: either $ab-c^2\neq 0$, in which case (a', b', c') must be proportional to (a, b, c) and the algebra is $L_5^+=\mathrm{u}(3)$ $(c^2-ab>0)$ or $L_5^-=\mathrm{gl}(3, R)$ $(c^2-ab<0)$; or else $ab-c^2=0$, and (a', b', c') must satisfy $ab'+a'b=2cc'$ and $L_5^0=[\mathrm{so}(3)\oplus\mathrm{u}(1)]\times P$ is a semidirect product, with P transforming as a 2-tensor under so(3) and as a one-dimensional representation under $u(1)$.

(vi) $\quad L_6=\langle G^{(1)}_\kappa, aT^{(2)}_\mu+bU^{(2)}_\mu+cG^{(2)}_\mu, aT^{(0)}_0+bU^{(0)}_0+cG^{(0)}_0, a'T^{(0)}_0+b'U^{(0)}_0+c'G^{(0)}_0\rangle.$

This forms a ten-dimensional Lie algebra if and only if $ab-c^2=0$ and $a'b+ab'=2cc'$. Then L_6 is a semidirect product $[\mathrm{so}(3)\oplus\mathrm{u}(1)]\times P$, with $\mathrm{u}(1)=\langle a'T^{(0)}_0+b'U^{(0)}_0+c'G^{(0)}_0\rangle$ and P the direct sum of a five-dimensional module $\langle aT^{(2)}_\mu+bU^{(2)}_\mu+cG^{(2)}_\mu\rangle$ (transforming as a 2-tensor under so(3) and a one-dimensional irrep under u(1)) and a one-dimensional module $\langle aT^{(0)}_0+bU^{(0)}_0+cG^{(0)}_0\rangle$.

It turns out that the space spanned by the three 0-tensors, one 1-tensor and one 2-tensor does not close under commutation. Hence, in the following we bring in two 2-tensors. The space spanned by one 1-tensor and two 2-tensors does not form a Lie algebra, unless we also introduce a 0-tensor:

(viii) $\quad L_7=\langle G^{(1)}_\kappa, aT^{(0)}_0+bU^{(0)}_0+cG^{(0)}_0, aT^{(2)}_\mu+bU^{(2)}_\mu+cG^{(2)}_\mu, a'T^{(2)}_\nu+b'U^{(2)}_\nu+c'G^{(2)}_\nu\rangle.$

This fourteen-dimensional vector space closes under commutation if $ab-c^2=0$ and $a'b+ab'=2cc'$; then $a'b'-c'^2\neq 0$. The Lie algebra L_7 is isomorphic to $L_7^+=\mathrm{su}(3)\times P$ or $L_7^-=\mathrm{sl}(3, R)\times P$, depending on the sign of $c'^2-a'b'$; $P=\langle aT^{(0)}_0+bU^{(0)}_0+cG^{(0)}_0, aT^{(2)}_\mu+bU^{(2)}_\mu+cG^{(2)}_\mu\rangle$ is a six-dimensional irreducible su(3) or sl(3, R) module.

(viii) $\quad L_8=\langle G^{(1)}_\kappa, a'T^{(0)}_0+b'U^{(0)}_0+c'G^{(0)}_0, a'T^{(2)}_\mu+b'U^{(2)}_\mu+c'G^{(2)}_\mu, aT^{(0)}_0+bU^{(0)}_0+cG^{(0)}_0, aT^{(2)}_\nu+bU^{(2)}_\nu+cG^{(2)}_\nu\rangle.$

This is a fifteen-dimensional Lie algebra if $ab-c^2=0$ and $a'b+ab'=2cc'$; then $a'b'-c'^2\neq 0$. It is again a semidirect product $L_8^+=\mathrm{u}(3)\times P$ or $L_8^-=\mathrm{gl}(3, R)\times P$ ($\pm$ is the sign of $c'^2-a'b'$), where P is a six-dimensional irreducible module spanned by the 0-tensor and 2-tensor with coefficients (a, b, c).

This exhausts all subalgebras of sp(3, R) containing so(3). The complete subalgebra chain is given in figure 1.

Until now only real linear combinations have been maintained. The above results are also valid for complex combinations (complex a, b, c, a', b', c') if the coefficients still satisfy

$$a'b-ab'(+\,\mathrm{cycl}),\ ab-c^2,\ a'b'-c'^2,\ a'b+ab'-2cc'\in R. \tag{19}$$

Then it can be checked by means of (10)-(13) that the structure constants are still real coefficients.

Note that the subalgebras $L_1, \ldots, L_8$ given in terms of coupled so(3) tensors can also be described in terms of the original cartesian products (1). For instance, by means of (1)-(5) and (14) one verifies that

$$\langle aT^{(0)}_0+bU^{(0)}_0+cG^{(0)}_0, aT^{(2)}_\mu+bU^{(2)}_\mu+cG^{(2)}_\mu\rangle = \langle (a+b+2c)Q_{ij}+(-a-b+2c)K_{ij}+\mathrm{i}(-a+b)S_{ij}\rangle. \tag{20}$$

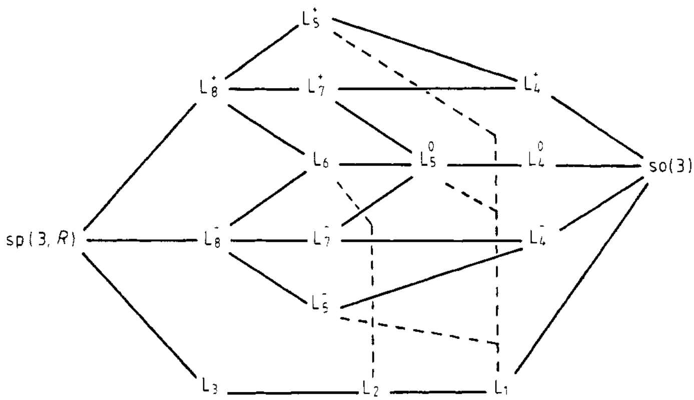

Figure 1. Subalgebra chain for sp(3, R)→so(3). The subalgebras L_i are described in the text. A line from left to right connecting L_i and L_j implies $L_i \supset L_j$; the lines connecting subalgebras from the upper part (starting from $L_8^{\pm}$) with subalgebras from the lower part (starting from L_3) are broken.

Hence

$$L = \langle L_{ij}, \alpha Q_{ij} + \beta K_{ij} + \gamma S_{ij} \rangle \tag{21}$$

forms a nine-dimensional Lie algebra of type L_5. The algebra $\langle L_{ij}, S_{ij}\rangle$ arises when $\alpha = \beta = 0$ and $\gamma \neq 0$, or equivalently when $a = -b \in iR$ and $c = 0$; it is clearly isomorphic to gl(3, R). Another realisation of gl(3, R) is the algebra $\langle L_{ij}, Q_{ij} - K_{ij} = x_i x_j - p_i p_j\rangle$; this Lie algebra is obtained for $\alpha = -\beta$ and $\gamma = 0$, or equivalently for $a = b \neq 0$ and $c = 0$. When $\alpha = \beta$ and $\gamma = 0$, or $a = b = 0$ and $c \neq 0$, the Lie algebra is $\langle L_{ij}, Q_{ij} + K_{ij} = x_i x_j + p_i p_j\rangle$ and is isomorphic to u(3). In general, one can check that $c^2 - ab$ has the same sign as $\alpha\beta - \gamma^2$, hence (21) is u(3), so(3)×$R^{[2]}$ or gl(3, R) when $\alpha\beta - \gamma^2$ is positive, zero or negative, respectively.

The realisation (10)–(13) given in terms of a pseudo p-boson π with angular momentum 1 can easily be extended to an arbitrary pseudo boson with angular momentum l. The Lie algebra that arises is the symplectic algebra sp($2l+1$, R). The commutation relations for such a realisation are simply a copy of (10)–(13) with $(-1)^{\kappa''+1}$ replaced by $(-1)^{\kappa''+l}$ and $\{\begin{smallmatrix} k & k' & k'' \\ 1 & 1 & 1 \end{smallmatrix}\}$ by $\{\begin{smallmatrix} k & k' & k'' \\ l & l & l \end{smallmatrix}\}$.

References

Edmonds A R 1957 *Angular Momentum in Quantum Mechanics* (Princeton, NJ: Princeton University Press)
Moshinsky M 1986 *Int. Conf. on Nuclear Structure, Reactions and Symmetries, Dubrovnik, Yugoslavia* (*June 1986*) (Singapore: World Scientific)
Rowe D J 1985 *Rep. Prog. Phys.* **48** 1419–80